Пределы и копределы

$X \in C$ (н) инициальным, если $\forall y \in C \quad \exists!\ x \to y$

терминальным, если $\forall y \in C \quad \exists!\ y \to x$

Set, Grp, Ring, Mon

def C категория, $X \in C$

$*$ $C_{/X}$ $\quad Ob \ni (y \to X)$

(слайс) $\quad$ over category

$$(y \xrightarrow{\varphi_y} x) \to (y' \xrightarrow{\varphi_{y'}} x)$$

$$\Leftrightarrow f : y \to y'$$

$$\begin{tikzcd} y \arrow[rr, "f"] \arrow[dr, "\varphi_y"'] & \arrow[d, phantom, "\Omega"] & y' \arrow[dl, "\varphi_{y'}"] \\ & x & \end{tikzcd}$$

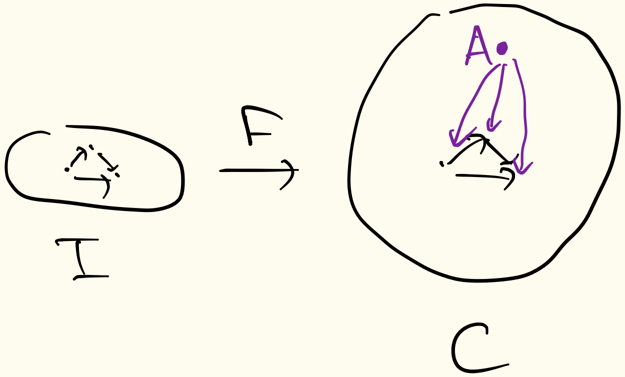
$C^{X/}$ $\quad$ —//— $\quad$ under category

$\boxed{ex}$ $\quad C = Set, \quad x = 1 = \{*\}$

$$C^{x/} = Set^{*/} \simeq Set_*$$

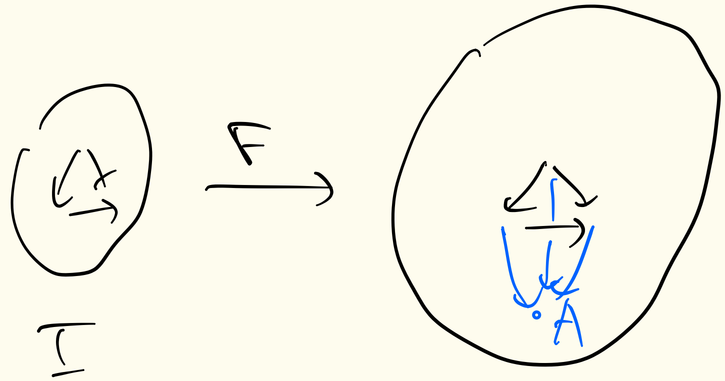
конус над диаграммой

$F: I \to C$



коконус над диаграммой

$F: I \to C$



def I, C категории, $A \in C$

$\underline{A}: I \to C$ $\quad i \mapsto A$

$(i \xrightarrow{\varphi} j) \mapsto 1_A$

$$\begin{tikzcd} & A \arrow[dr, "id"] & \\ A \arrow[ur, "id"] \arrow[rr, "id"'] & & A \end{tikzcd}$$

def $F: I \to C$ конусом над F н (A, α), где
i

$A \in C$, $\quad \alpha: \underline{A} \Rightarrow F$

$(\alpha_i: A \to F(i) \quad \forall i \in I)$

$$\begin{tikzcd} & A \arrow[dl, "\alpha_i"'] \arrow[dr, "\alpha_j"] & \\ F(i) \arrow[rr, "F(\varphi)"'] & & F(j) \end{tikzcd}$$

$\forall i \xrightarrow{\varphi} j$ в I

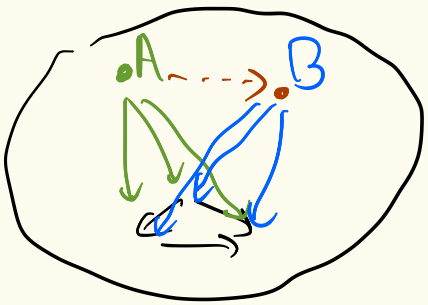
$$\begin{tikzcd} A \arrow[r, "id"] \arrow[d, "\alpha_i"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, "\alpha_j"] \\ F(i) \arrow[r] & F(j) \end{tikzcd}$$

Коконус над F это (A, α), где $A \in C$, $\alpha : F \Rightarrow \underline{A}$

def $F : I \to C$ $\quad (A, \alpha), (B, \beta)$ конусы над F

$$(A,\alpha) \xrightarrow{?} (B,\beta) \iff A \xrightarrow{f} B \text{ т.ч. } \forall i \in I$$

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\alpha_i"'] & \arrow[d, phantom, "\circlearrowleft"] & B \arrow[dl, "\beta_i"] \\
& F(i) &
\end{tikzcd}$$

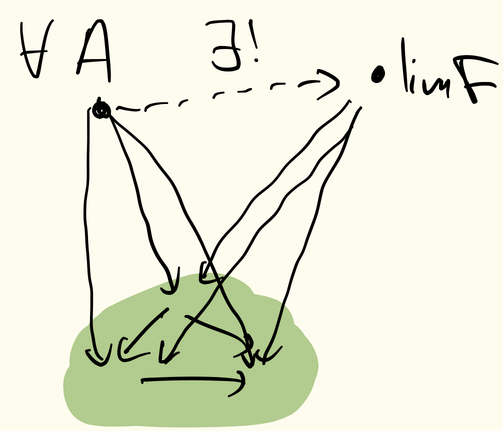


C

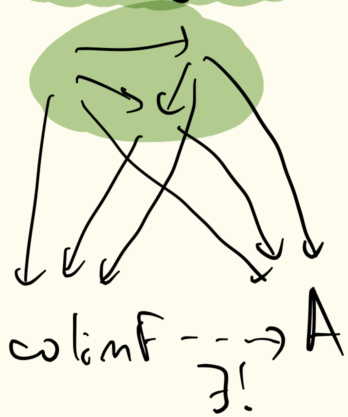
def $F : I \to C$ функтор

пределом F (н) терминальный конус над F

$(\lim F, \alpha)$



Копределом F @ инициальный коконус $(\operatorname{colim} F, \alpha)$



Утв если $\exists \lim F$, то автомат. (!)

$$\left(\forall\ (\lim F', \alpha')\ \exists!\ (\lim F', \alpha') \cong (\lim F, \alpha)\right)$$

Пример 1

$I = \varnothing \qquad \varnothing \xrightarrow{F} C$

категория конусов над F = C

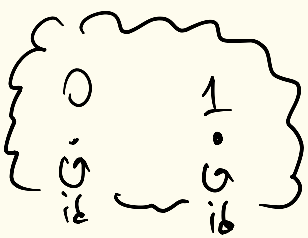
коконусов = -//-

$\lim F$ = терминальный объект C

$\operatorname{colim} F$ = инициальный объект C

Пример 2

$I = \operatorname{discr}(2) \qquad F : I \to C$



$\operatorname{Fun}(I, C) = C \times C$

$F(0) = X_0, \quad F(1) = X_1$

$$\begin{tikzcd}
& \lim F = X_0 \times X_1 \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
X_0 & & X_1
\end{tikzcd}$$

(π_i, pr_i)

$$\begin{tikzcd}
& \lim F \arrow[dl] \arrow[dr] & \\
X_0 \arrow[r, phantom, "\circlearrowleft"] & & X_1 \arrow[l, phantom, "\circlearrowright"] \\
& A \arrow[ul, "\forall"] \arrow[ur, "\forall"'] \arrow[uu, dashed, "\exists!"] &
\end{tikzcd}$$

$\forall$

Наблюдение $A \in C, \ X_0, X_1 \in C$

$$\{A \xrightarrow{\varphi} X_0 \times X_1\} \longleftrightarrow \left\{(\varphi_0, \varphi_1) \;\middle|\; \begin{array}{l} \varphi_0 : A \to X_0 \\ \varphi_1 : A \to X_1 \end{array}\right\}$$

$$\| $$

$$\mathrm{Hom}_C(A, X_0 \times X_1)$$

$$\begin{tikzcd}
& & X_0 \\
A \arrow[r, "\varphi"] \arrow[urr, bend left, "\varphi_0"] \arrow[drr, bend right, "\varphi_1"'] & X_0 \times X_1 \arrow[ur, "p_0"] \arrow[dr, "p_1"'] & \\
& & X_1
\end{tikzcd}$$

Общий факт

$$\mathrm{Hom}_C(A, \lim F) \cong \mathrm{Hom}_{\mathrm{Fun}(I,C)}(\underline{A}, F)$$

6 Set $\quad X_0 \times X_1 = \{ (x_0, x_1) \mid x_i \in X_i \}$

$p_i(x_0, x_1) = x_i \qquad p_i : X_0 \times X_1 \to X_i$

$$\begin{tikzcd}
 & X_0 \times X_1 \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
X_0 \arrow[r, phantom, "\circlearrowleft"] & \exists! \arrow[r, phantom, "\circlearrowleft"] & X_1 \\
 & A \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, dashed, bend left=0] & \\
 & \forall &
\end{tikzcd}$$

$\boxed{\text{ex}}$ $(X, \leq) \in \text{Poset} \subseteq \text{Cat}$

$x \xrightarrow{!} y \iff x \leq y$

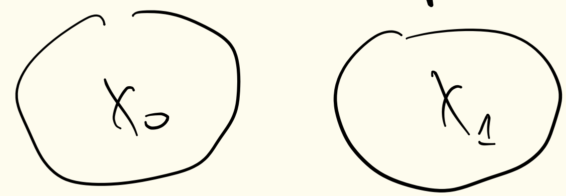
$x_0, x_1 \in X \qquad \inf(x_0, x_1) = x_0 \wedge x_1 = \boxed{x_0 \times x_1} \in X$

$$\begin{tikzcd}
 & x_0 \times x_1 \arrow[dl] \arrow[dr] & \\
x_0 & \exists! & x_1 \\
 & \forall a \arrow[ul] \arrow[ur] \arrow[uu, dashed] &
\end{tikzcd}$$

$$\begin{tikzcd}
X_0 \arrow[d] \arrow[drr] & & X_1 \arrow[dll] \arrow[d, "\forall"] \\
\operatorname{colim} F \arrow[rr, dashed, "\exists!"'] & & A \ \forall
\end{tikzcd}$$

$$\operatorname{colim} F = X_0 \sqcup X_1$$

синий красный



$$X_0 \sqcup X_1 = X_0 \times \{0\} \cup X_1 \times \{1\}$$

$$\begin{tikzcd}
& X_0 \sqcup X_1 & \\
X_0 \arrow[ur] & & X_1 \arrow[ul]
\end{tikzcd}$$

$$\operatorname{Hom}_{\operatorname{Fun}(I,C)}(F, \underline{A}) = \operatorname{Hom}_C(\operatorname{colim} F, A)$$

$$\operatorname{Hom}_C(X_0, A) \times \operatorname{Hom}_C(X_1, A) = \operatorname{Hom}_C(X_0 \sqcup X_1, A)$$

$$\begin{tikzcd}
 & X_0 \sqcup X_1 \arrow[dd, dashed, "\exists!"] & \\
X_0 \arrow[ur] \arrow[dr, "\forall"'] & & X_1 \arrow[ul] \arrow[dl, "\forall"] \\
 & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X_0 \times X_1 \arrow[dl] \arrow[dr] & \\
X_0 & & X_1 \\
 & A \arrow[ul, "\forall"] \arrow[ur, "\forall"'] \arrow[uu, dashed, "\exists!"] &
\end{tikzcd}$$

аналогично

$I = \mathrm{discr}(\Lambda)$ $\qquad$ $X_\lambda = F(\lambda)$

$$\begin{tikzcd}
\lim F = \prod\limits_{\lambda \in \Lambda} X_\lambda \arrow[d, "p_\lambda"'] \\
X_\lambda
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{colim} F = \coprod\limits_{\lambda \in \Lambda} X_\lambda \\
X_\lambda \arrow[u, "\iota_\lambda"]
\end{tikzcd}$$

(или in_λ)

уравнители и коуравнители

(эквалайзеры коэквалайзеры)

$$I \quad \begin{tikzcd} 0 \arrow[loop above, "id"] \arrow[r, shift left] \arrow[r, shift right] & 1 \arrow[loop above, "id"] \end{tikzcd}$$

$I \xrightarrow{F} C$? $\equiv$ выбор в C двух

$f, g \in \mathrm{Mor}\, C$

$\mathrm{dom}(f) = \mathrm{dom}(g)$

$\mathrm{cod}(f) = \mathrm{cod}(g)$

$eq(f,g) = \lim F$

$$\begin{tikzcd} eq(f,g) \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ \forall A \arrow[u, dashed, "\exists!"] \arrow[ur] \arrow[u, phantom, shift left=6, "\circlearrowright"] & & \end{tikzcd}$$

конус?

это морфизм $\varphi: A \to X$ т.з.

$f \circ \varphi = g \circ \varphi$

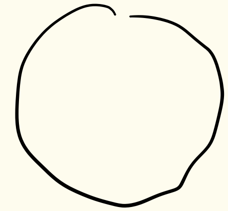
в Set $eq(f,g) = \{ x \in X \mid f(x) = g(x) \} \subseteq X$

$e: eq(f,g) \longrightarrow X$

$x \longmapsto x$

$x^2+y^2=1$

$$X \subseteq \mathbb{R}^2 \underset{(x,y)\mapsto x^2+y^2}{\overset{(x,y)\mapsto 1}{\rightrightarrows}} \mathbb{R}$$



$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r] \arrow[dr] \arrow[r, phantom, bend right, "\circlearrowleft"'] & \operatorname{coeq}(f,g)=\operatorname{colim} F \arrow[d, dashed, "\exists!"] \\
 & & A \quad \forall
\end{tikzcd}$$

Упр

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \longrightarrow Y/\sim$$

$\forall x \in X$

$f(x) \sim g(x)$

def Морфизм $X \xrightarrow{f} Y$ н **регулярным мономорфизмом**, если

$\exists Z$ и $Y \rightrightarrows Z$ т.ч. $X \xrightarrow{f} Y \rightrightarrows Z$ диаграмма эквал.

Утв f рег. моно $\Rightarrow$ f моно

Pf

$$\begin{tikzcd} & X \arrow[r, "f"] & Y \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & Z \\ A \arrow[ur, shift left, "\varphi"] \arrow[ur, shift right, "\psi"'] \arrow[urr, "f\circ\varphi = f\circ\psi"'] & & & \end{tikzcd}$$

$\blacksquare$

Утв f расщепл. моно $\Rightarrow$ f регулярный

Pf

$$\begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "\circlearrowleft"] & \\ X \arrow[ur, "f"] \arrow[rr, "id_X"'] & {} & X \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "id_Y"] \arrow[r, shift right, "f\circ g"'] \arrow[dr, "g"'] & Y \\ & & X \arrow[u, "f"'] \\ Z \arrow[uu, "\exists\, g\circ a"] \arrow[uur, "a"'] & & \end{tikzcd}$$

$\blacksquare$

split mono $\subseteq$ regular mono $\subseteq$ (ordinary) mono

Пример 3

$$I \quad \boxed{\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}} \xrightarrow{F} C$$

$$\begin{tikzcd} X \times_Z Y \arrow[rrd, bend left] \arrow[rdd, bend right] & & \\ & A \arrow[ul, dashed, "\exists!"'] \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$\lim F = X \times_Z Y$ (пулбэк X и Y над Z)

Утв $X \times_Z Y = eq\left(X \times Y \overset{f \circ \pi_X}{\underset{g \circ \pi_Y}{\rightrightarrows}} Z \right)$

Pf

$$\begin{tikzcd} A \arrow[r, dashed, "\exists!"] \arrow[drr] \arrow[drrr, bend left] \arrow[ddrr, bend right] & eq \arrow[dr] \arrow[drr] \arrow[ddr] & & \\ & & X \times Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] & X \arrow[d, "f"] \\ & & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$\square$

Пример

$$\begin{tikzcd} X\times_Z Y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

$$X\times_Z Y = \{(x,y)\in X\times Y \mid f(x)=g(y)\}$$

Нотация

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

квадрат н. декартовым, если

$$(A, \{A\to X, A\to Y\}) \cong X\times_Z Y$$

$$I \xrightarrow{F} C \xleftarrow{G} J$$

$\rightsquigarrow$ $F\downarrow G$

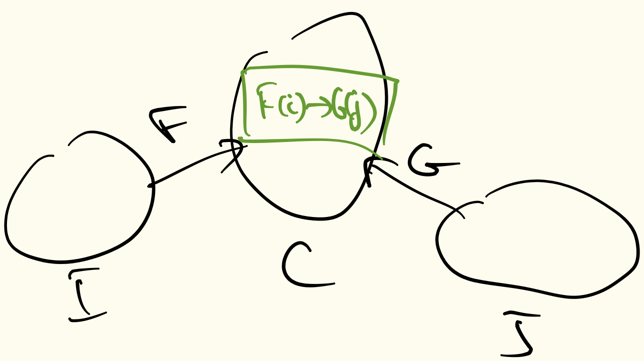
комма-категория

Ob это тройки $(i\in I, j\in J, F(i)\xrightarrow{f} G(j))$

морфизмы $(i,j,f)\to(\tilde i,\tilde j,\tilde f)$ это

пары морфизмов $i\xrightarrow{\varphi}\tilde i$, $j\xrightarrow{\psi}\tilde j$ т.ч.

$$\begin{tikzcd} F(i) \arrow[r, "f"] \arrow[d, "F(\varphi)"'] \arrow[dr, phantom, "\circlearrowleft"] & G(j) \arrow[d, "G(\psi)"] \\ F(\tilde i) \arrow[r, "\tilde f"'] & G(\tilde j) \end{tikzcd}$$



Конусы над F

$$\left(\begin{tikzcd} & \mathrm{Fun}(I, C) & \\ C \arrow[ur, "\mathrm{const}"] & & * \arrow[ul, "F"'] \end{tikzcd}\right) \qquad \mathrm{const} \downarrow F$$

$$\boxed{\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & X \\ Y & \end{tikzcd}}$$

F

пушаут

$$\operatorname{colim} F = X \sqcup_A Y = \operatorname{coeq}\left(A \rightrightarrows X \sqcup Y\right)$$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

кодекартов

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d] \arrow[ddr, bend left, "\forall"] & \\ Y \arrow[r] \arrow[drr, bend right, "\forall"'] & X \sqcup_A Y \arrow[dr, dashed, "\exists!"] & \\ & & Z \end{tikzcd}$$

$$\frac{X \sqcup Y}{\{f(a) \sim g(a)\ \forall a \in A\}}$$

Утв I есть инициальный $i_0 \in I$

$\Rightarrow \forall F: I \to C \quad \exists \lim F = F(i_0)$ Упр

Th C кат. СУР:

1. в C есть пределы произвольных (малых) диаграмм

$\left(\forall \underset{\text{малой}}{I},\ I \xrightarrow{F} C \quad \exists \lim F \in C\right)$

2. в C есть произвольные малые произведения и уравнители

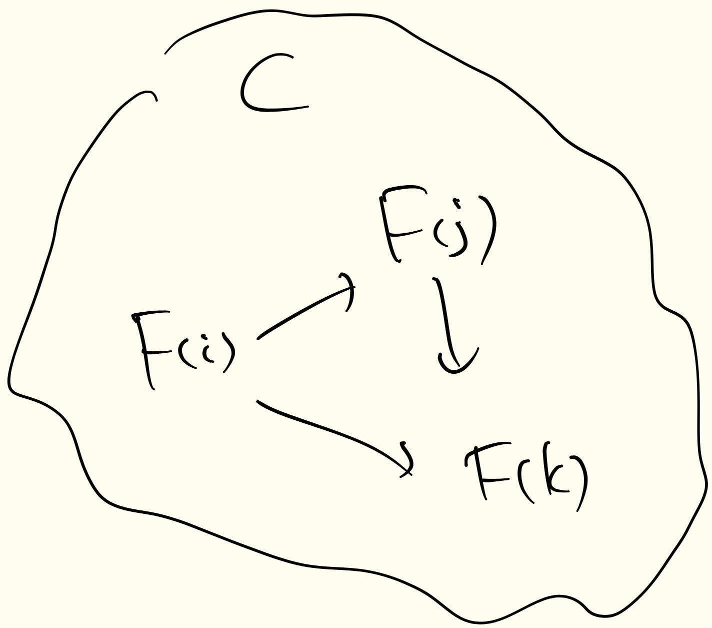
Pf $\boxed{1 \Rightarrow 2}$ Ясно

$\boxed{2 \Rightarrow 1}$

$$I \xrightarrow{F} C$$

Малая

$$\lim F = eq\left(\prod_{i\in I} F(i) \overset{\Phi}{\underset{\Psi}{\rightrightarrows}} \prod_{\varphi: i\to j} F(j)\right)$$



$$\forall\, (\varphi : i \to j) \in Mor(I)$$

$$\Phi_\varphi : \prod_{i\in I} F(i) \xrightarrow{\pi_i} F(i) \xrightarrow{F(\varphi)} F(j)$$

$$\Psi_\varphi : \prod_{i\in I} F(i) \xrightarrow{\pi_j} F(j)$$

структура конуса:

$$\begin{tikzcd} \lim F \arrow[r, hook] \arrow[dr, "\alpha_i"'] & \prod\limits_{i\in I} F(i) \arrow[d, "\pi_i"] & i \in I \\ & F(i) & \end{tikzcd}$$

$$\begin{tikzcd} & \lim F \arrow[dl, "\alpha_i"'] \arrow[dr, "\alpha_j"] \arrow[d, phantom, "\circlearrowleft"] & \\ F(i) \arrow[rr, "F(\varphi)"'] & & F(j) \end{tikzcd}$$

$\square$

$$I \xrightarrow{F} Set$$

$$\lim F = \left\{ (x_i)_{i\in Ob I} \in \prod_{i \in Ob I} F(i) \;\middle|\; \begin{array}{l} \forall (\varphi : i \to j) \in Mor(I) \\ F(\varphi) : F(i) \to F(j) \quad F(\varphi)(x_i) = x_j \end{array} \right\}$$

$$\lim F \xrightarrow{\alpha_i} F(i)$$

для colim — Упр

$$C \underset{R}{\overset{L}{\rightleftarrows}} D \quad \perp$$

$$X \underset{\varphi^*}{\overset{\varphi}{\rightleftarrows}} Y$$

гильбертовы

$$\langle \varphi x, y\rangle_Y = \langle x, \varphi^* y\rangle_X$$

L левый сопр. к R,

если $\mathrm{Hom}_D(Lx, y) = \mathrm{Hom}_C(x, Ry)$

(Кан)

$$C^{op}\times D \longrightarrow Set$$

$$(x,y) \longmapsto \ldots$$

I малая кат.

$$C \xrightarrow{const} \mathrm{Fun}(I, C)$$

$$A \longmapsto \underline{A}$$

colim ⊣ const ⊣ lim

$$\mathrm{Hom}_C(\operatorname{colim} F, A) = \mathrm{Hom}_{\mathrm{Fun}(I,C)}(F, \underline{A})$$

$$\mathrm{Hom}_{\mathrm{Fun}(I,C)}(\underline{A}, F) = \mathrm{Hom}_C(A, \lim F)$$